Christ Church
Grammar School

2018
TEST 5

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Student name SOLUTIONS.

Teacher name ______________________

Time and marks available for this section

Reading Time: 2 minutes
Working time for this section: 15 minutes
Marks available: 15 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(4 marks)**

Find the gradient of the curve $xy^3 = 5\ln y$ at the point (0,1).

$$xy^3 = 5\ln y$$

$$y^3 + x.3y^2\frac{dy}{dx} = 5.\frac{1}{y}\frac{dy}{dx} \quad ✓$$

$$y^3 = \frac{5}{y}\frac{dy}{dx} - 3xy^2\frac{dy}{dx}$$

$$y^3 = \frac{dy}{dx}\left(\frac{5}{y} - 3xy^2\right) \quad ✓$$

$$y^3 = \frac{dy}{dx}\left(\frac{5-3xy^3}{y}\right)$$

$$\frac{y^4}{5-3xy^3} = \frac{dy}{dx} \quad ✓$$

when $x=0$, $y=1$

$$\frac{1}{5-0} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{5} \quad ✓$$

can get 4th mark for $\frac{dy}{dx}$ if $\frac{dy}{dx}$ is incorrect.

F.T marks for incorrect differentiation at step 1.

Question 2 **(6 marks)**

A particle moves in a straight line with acceleration given by $\ddot{x} = -9x$, where x is the position of the particle at time t. The particle's initial position and velocity are $x(0) = 0$ and $\dot{x}(0) = 4$. Find:

(a) the period and amplitude of the motion. (3 marks)

$\ddot{x} = -9x$
$= -n^2 x$
$\therefore n = 3$ ✓

(period) $T = \frac{2\pi}{n}$
$= \frac{2\pi}{3}$ ✓

FT. if 'n' is incorrect. ✓

$v^2 = n^2(A^2 - x^2)$
$\dot{x}(0) = 4 = v$
$x(0) = 0$
$\therefore 4^2 = 3^2(A^2 - 0^2)$
$16 = 9A^2$
$\frac{16}{9} = A^2$ ✓
$\therefore A = \frac{4}{3}$ (amplitude) ✓

or Assume
$x = a\sin(kt + \alpha)$
$x(0) = 0 \therefore \alpha = 0$
$x = a\sin(3t)$
$\dot{x} = 3a\cos(3t)$
$4 = 3a\cos 0$
$\therefore a = \frac{4}{3}$

(b) the particle's position, velocity and acceleration at time t. (3 marks)

$x = A\sin(nt + c)$
$= \frac{4}{3}\sin(3t + c)$ when $t = 0, x = 0 \therefore c = 0$

$x = \frac{4}{3}\sin(3t)$ ✓

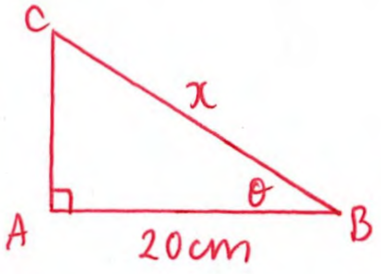
$\dot{x} = 4\cos(3t)$ ✓

$\ddot{x} = -12\sin(3t)$ ✓

can get 3/3 if correct process but incorrect 'n' value from part (a).

Question 3 **(5 marks)**

Triangle ABC is right angled at A, and AB = 20 cm. The angle ABC increases at a constant rate of 1° per minute. At what rate is BC changing at the instant when angle ABC is 30°?



Let $\angle ABC = \theta$, $BC = x$.

We want $\frac{dx}{dt}$ when $\theta = 30° = \frac{\pi}{6}^c$

$\frac{d\theta}{dt} = 1° = \frac{\pi}{180}$ radians/min ✓

$\cos\theta = \frac{20}{x}$

$x = \frac{20}{\cos\theta}$

$= 20(\cos\theta)^{-1}$

$\frac{dx}{d\theta} = -20(\cos\theta)^{-2} \cdot -\sin\theta$

$= \frac{+20\sin\theta}{\cos^2\theta}$ ✓

$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{\cancel{20}^{1}\sin\theta}{\cos^2\theta} \times \frac{\pi}{\cancel{180}_{9}}$ ✓ For correct rule + simplifying

$= \frac{\pi\sin\theta}{9\cos^2\theta}$

when $\theta = \pi/6$

$\frac{dx}{dt} = \frac{\pi\sin\pi/6}{9(\cos\pi/6)^2}$ ✓ for subst $\theta = \pi/6$.

$= \frac{\pi \times \frac{1}{2}}{9\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{\frac{\pi}{2}}{9 \times \frac{3}{4}}$

$= \frac{\pi}{2} \times \frac{4}{27}$

$= \frac{2\pi}{27}$ cm/min ✓

If students leave θ in degrees max mark $\frac{4}{5}$.

If θ is in degrees $\frac{dx}{dt} = 13\frac{1}{3}$.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Christ Church Grammar School

2018
TEST 5

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Student name SOLUTIONS.

Teacher name ______________________

Time and marks available for this section

Reading time before commencing work: 3 minutes
Working time for this section: 30 minutes
Marks available: 30 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

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Question 4 **(6 marks)**

When used in a torch, the lifetime of a single 9 Volt C size battery was observed to be normally distributed with a mean of μ hours and a standard deviation of σ hours.

A student bought 30 boxes of these batteries, with 36 batteries in each box, and calculated the average lifetime for the batteries in each box. The mean of the averages was 30.45 hours and the standard deviation of the averages was 0.38 hours.

(a) Use this information to determine estimates for μ and σ. (3 marks)

$\mu = 30.45\ h$ ✓

recognise $\mu \simeq$ mean of averages

$$\sigma = \frac{\sigma_p}{\sqrt{n}}$$

$$0.38 = \frac{\sigma_p}{\sqrt{36}}$$ ✓

recognise 0.38 is st.dev of averages.

$\sigma_p \approx 2.28$ hrs (2dp) ✓

any rounding accepted.

FT if step 2 is incorrect.

(b) The batteries in one of the boxes lasted for a total of 1094 hours. Use this sample of 36 batteries to construct a 99% confidence interval for the lifetime of this type of battery. (3 marks)

$$\bar{x} = \frac{1094}{36}$$

$\simeq 30.389\ h$ (3dp) ✓ any rounding accepted

CI error : 2.576×0.38

$= 0.979$ ✓ or $\mu \pm 2.576 \times \sigma$

$30.389 \pm 2.576 \times 0.38$

$\therefore\ 30.389 \pm 0.979$

CI $29.41 \leq h \leq 31.37$ ✓

values must be with original limits + this will depend on dp.

Question 5 **(8 marks)**

A tank contains 30 litres of a solution of a chemical in water. The concentration of the chemical is reduced by running pure water into the tank at a rate of 1 litre per minute and allowing the solution to run out of the tank at a rate of 2 litres per minute. The tank contains x litres of the chemical at time t minutes after the dilution starts.

(a) Show that $\frac{dx}{dt} = \frac{-2x}{30-t}$ (2 marks)

At t mins $V = 30 - t$ (out @ 2L/min and in @ 1L/min)

At t mins, fraction of solution which is chemical is $\frac{x}{30-t}$

$\therefore \frac{dx}{dV} = \frac{x}{30-t}$ ✓

rate of flow of chemical out of tank is 2 ($\therefore -2$) $\frac{dV}{dt} = -2$ ✓

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} = \frac{x}{30-t} \cdot -2 = \frac{-2x}{30-t}.$$

(b) Find the general solution of this differential equation, in terms of x_0, where x_0, is the initial amount of chemical in the tank. (5 marks)

$$\int \frac{dx}{-2x} = \int \frac{1}{30-t}\, dt \quad ✓ \quad \text{set up of } \int$$

$$\Rightarrow -\tfrac{1}{2}\ln|x| = -\ln|30-t| + c$$
$$\ln|x| = 2\ln|30-t| + c \quad ✓ \quad \text{simplified}$$

when $t=0$ $x = x_0$

$$\therefore \ln x_0 = 2\ln|30-0| + c$$

$$\Rightarrow c = \ln x_0 - \ln 30^2$$
$$= \ln\left(\frac{x_0}{900}\right) \quad ✓$$

$$\ln x = \ln(30-t)^2 + \ln\left(\frac{x_0}{900}\right)$$

$$\ln x = \ln\left(\frac{x_0}{900}(30-t)^2\right) \quad ✓ \quad \text{simplified}$$

$$x = \frac{x_0}{900}(30-t)^2 \quad ✓ \quad \text{eqn as } x =$$

or $$x = \frac{x_0(30-t)^2}{900}$$

Question 5 continued

(c) Find the fraction of the original chemical still in the tank after 20 minutes.

(1 mark)

When $t = 20$ $x = \frac{x_0}{900}(30-20)^2$

$= \frac{1}{9}x^0$

$\therefore \frac{1}{9}$th of original amount.

Question 6 **(7 marks)**

A population grows according to the differential equation:

$$\frac{dP}{dt} = 0.025P\left(1 - \frac{P}{1000}\right), \qquad 0 < P < 1000$$

where P is the population at time t. When $t = 0, P = 20$.

(a) Find the population P at time t, leaving answers in exact form where necessary. (5 marks)

$$\frac{dP}{dt} = 0.025P\left(1 - \frac{P}{1000}\right)$$

$$\frac{dP}{dt} = \frac{P(1000-P)}{40000}$$

$$\int \frac{40000}{P(1000-P)}\,dP = \int dt \quad ✓$$

$$\int\left(\frac{40}{P} + \frac{40}{1000-P}\right)dP = t \quad ✓$$

using clas pad or by hand

$$\frac{A}{P} + \frac{B}{1000-P} = \frac{40000}{P(1000-P)}$$

$$\Rightarrow 40000 = (1000-P)A + BP$$

$P=0 \quad \therefore 40000 = 1000A$, $A = 40$

$P=1000 \quad \therefore 40000 = 1000B$, $B = 40$

$$40\int\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP = t$$

$$40(\ln P - \ln|1000-P|) + c = t$$

$$40\ln\left(\frac{P}{1000-P}\right) + c = t$$

$$\ln\left(\frac{P}{1000-P}\right) = \frac{t-c}{40}$$

$$\frac{P}{1000-P} = e^{(t-c)/40}$$

Let $A = e^{-c/40}$

$$\frac{P}{1000-P} = Ae^{t/40} \quad ✓$$

when $t=0$, $P=20$

$$\frac{20}{980} = A = \frac{1}{49} \quad ✓$$

$$P = \frac{1}{49}e^{t/40}(1000-P)$$

$$49P + e^{t/40}P = 1000e^{t/40}$$

$$P = \frac{1000e^{t/40}}{49 + e^{t/40}} = \frac{1000}{1 + 49e^{-t/40}} \quad ✓$$

$$\frac{dP}{dt} = ay - by^2$$

$$\Rightarrow y = \frac{a}{b + ce^{-at}}$$

$$\therefore \frac{dP}{dt} = \frac{1}{40}P - \frac{P^2}{40000} \quad ✓$$

$$\therefore a = \frac{1}{40} \quad b = \frac{1}{40000} \quad ✓$$

$$\Rightarrow P = \frac{\frac{1}{40}}{\frac{1}{40000} + ce^{-1/40t}} \times \frac{40000}{40000} \quad ✓$$

$$= \frac{1000}{1 + 40000ce^{-1/40t}}$$

When $t=0$, $P=20$

$$20 = \frac{1000}{1 + 40000ce^{0}} \quad ✓$$

$$\therefore c = \left(\frac{1000}{20} - 1\right) \div 40000 = \frac{49}{40000}$$

$$\therefore P = \frac{1000}{1 + 49e^{-t/40}} \quad ✓$$

Question 6 continued

(b) Find the population when the rate of growth is at a maximum. (2 marks)

$$\frac{dP}{dt} = \frac{1000P - P^2}{40000}$$

$$\frac{d^2P}{dt^2} = \frac{1000 - 2P}{40000} \cdot \frac{dP}{dt}$$

let $\frac{d^2P}{dt^2} = 0$

$0 = 1000 - 2P$

$\therefore P = 500$ ✓✓

or graph $\frac{dP}{dt} = \frac{1000P - P^2}{40000}$

(or $\frac{dP}{dt} = 0.025P\left(1 - \frac{P}{1000}\right)$)

Find TP $(500, 6.25)$

$\therefore$ max at $P = 500$.

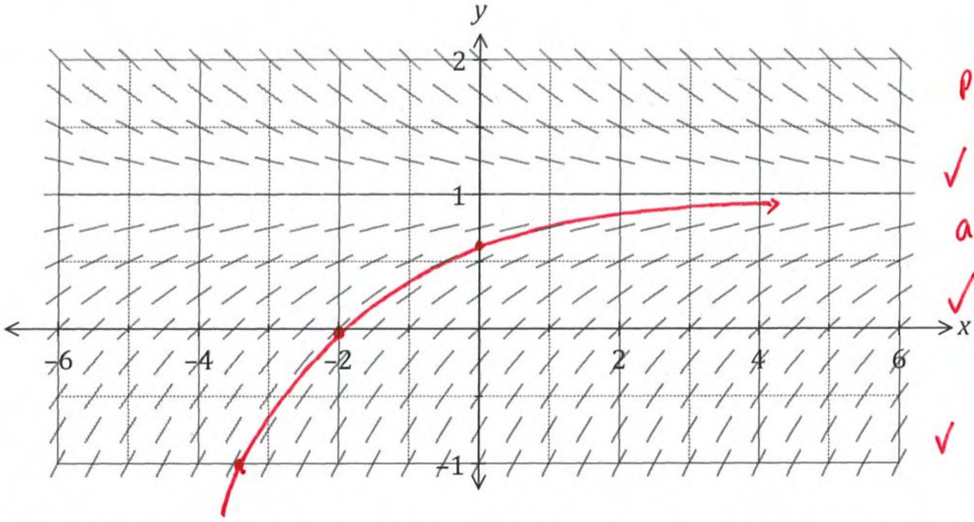
no working required for 2marks.

or $\frac{dP}{dt}$ is a quadratic $\therefore$ max $\frac{1}{2}$ way between solutions of 0 & 1000.

So $P = 500$.

Question 7 **(9 marks)**

A first-order differential equation has a slope field as shown below. The solution of the equation passes through the point $P(-2, 0)$, where the value of the slope is 0.5.



Part (b)

✓ through $P(-2,0)$ and close to $(-3.5,-1)$

✓ close to y-int $(0, 0.65)$

✓ Asymptote at $y=1$.

(a) The general differential equation for the slope field is of the form below, where a and b are constants:

$$\frac{dy}{dx} = a(y + b)$$

Derive the solution to this equation that passes through P in the form $y = f(x)$. (6 marks)

$\frac{dy}{dx} = 0$ when $y = 1$ $\quad 0 = a(1+b) \therefore b = -1$ ✓

$0.5 = a(0-1)$

$a = -0.5$ (or $-\frac{1}{2}$) ✓

If use CAS max $\frac{3}{6}$

$\frac{dy}{dx} = -\frac{1}{2}(y-1)$

$\int \frac{1}{y-1}\,dy = \int -\frac{1}{2}\,dx.$

$\ln|y-1| = -\frac{1}{2}x + c$ ✓

$P(-2, 0)$ $\quad \ln|-1| = -\frac{1}{2}(-2) + c \quad \therefore c = -1$ ✓

$y < 1 \Rightarrow \ln(1-y) = -\frac{1}{2}x - 1$ ✓

$1 - y = e^{-1/2x-1}$

$y = 1 - e^{-1/2x-1}$ ✓

(b) Sketch the solution of the equation that passes through $P(-2, 0)$, where the value of the slope is 0.5, onto the graph at the top of the page. (3 marks)

End of questions

Additional working space

Question number:______________

Additional working space

Question number:_____________